Noise-Free Measurement of Harmonic Oscillators with Instantaneous Interactions

M. França Santos,[1,*] G. Giedke,[2,†] and E. Solano[2,3,4,‡]

[1]*Dep. de Física, Universidade Federal de Minas Gerais, Belo Horizonte, 30161-970, MG, Brazil*
[2]*Max-Planck-Institut für Quantenoptik, Hans-Kopfermann-Str. 1, D-85748 Garching, Germany*
[3]*Physics Department, ASC, and CeNS, Ludwig-Maximilians-Universität, Theresienstrasse 37, 80333 Munich, Germany*
[4]*Sección Física, Departamento de Ciencias, Pontificia Universidad Católica del Perú, Apartado Postal 1761, Lima, Peru*

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We present a method of measuring the quantum state of a harmonic oscillator through instantaneous probe-system selective interactions of the Jaynes-Cummings type. We prove that this scheme is robust to general decoherence mechanisms, allowing the possibility of measuring fast-decaying systems in the weak-coupling regime. This method could be applied to different setups: motional states of trapped ions, microwave fields in cavity/circuit QED, and even intra-cavity optical fields.

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Measuring the quantum state of a harmonic oscillator, or, equivalently, its associated Wigner function [1], is a fundamental task of quantum physics. Some proposals allow a direct measurement, like propagating optical fields tested with homodyning techniques [2]. Others, due to problems of accessibility, require indirect measurement schemes via interaction with a probe. This is the case of microwave fields in 3D cavities [3], circuit cavity QED with superconducting qubits [4], or the motion of trapped ions [5]. Different as they are, known techniques share a common problem: the noisy action of decoherence due to the probe-system *finite interaction times*.

Recently, the reconstruction of a Wigner function in microwave cavity QED (CQED) was successfully realized [6] with the aid of a dispersive probe-system interaction [7]. Unfortunately, dispersive coupling is known to be slow and the required interaction time allows decoherence processes to disturb the measurement. In Ref. [8], a resonant method was proposed, but the Wigner reconstruction depends on the possibility of monitoring a few Rabi cycles, adding up to long probe observation periods. In the case of trapped ions, the measurement techniques are quite similar and a recent experiment [9] made use of numerical integration over several Rabi cycles to achieve the goal. These long interaction times are particularly harmful in the case of fast-decaying systems. For example, state reconstruction of intracavity optical fields has not been experimentally attempted, to our knowledge, due to their weak coupling with atomic probes.

In this paper, we present a method to measure the quantum state of a harmonic oscillator through *instantaneous* probe-system interactions [10, 11], preventing decoherence from disturbing the measurement. The harmonic oscillator is allowed to interact with a two-level probe for an arbitrarily short time via a selective interaction [12, 13, 14] in the Jaynes-Cummings (JC) model [15]. The information is then collected from the second time derivative of the probe population at *zero interaction time*. The scheme permits to measure the population field distribution and, with the support of coherent displacements, the associated Wigner and Q functions at any point in phase space with arbitrarily small influence of decoherence. From this data the full Wigner function can be reconstructed by a simple fit as in [6] or by more sophisticated techniques [16], e.g., involving maximum likelihood estimation [17] and taking into account the imperfections of the measurement process [2].

Typically, a selective interaction can be built when a three-level probe, driven by a classical and a quantized field, is reduced to two metastable states after adiabatic elimination of the third level, allowed by a large detuning Δ. The associated Hamiltonian reads [12, 13]

$$\hat{H}_{\rm eff} = \hbar\frac{\Omega_1^2}{\Delta}|g\rangle\langle g| + \hbar\frac{\Omega_2^2}{\Delta}\hat{a}^\dagger\hat{a}|e\rangle\langle e| + \hbar\frac{\Omega_1\Omega_2}{\Delta}\big(|g\rangle\langle e|\hat{a}^\dagger + |e\rangle\langle g|\hat{a}\big). \quad (1)$$

Here, $\{|g\rangle, |e\rangle\}$ are the (metastable) ground and excited states of the two-level probe, $\{\hat{a}, \hat{a}^\dagger\}$ are the harmonic oscillator annihilation and creation operators, respectively, and $\Omega \equiv \Omega_1\Omega_2/\Delta$ is the effective JC coupling strength. The first and second terms on the r.h.s. of Eq. (1) are AC-Stark shifts associated with each of the probe levels, the second one depending on the number of oscillator excitations. This means that any effort at tuning the JC coupling to resonance will succeed only for a selected JC doublet $\mathcal{H}_N : \{|g\rangle|N\rangle, |e\rangle|N-1\rangle\}$, leaving all other doublets, for which $n \neq N$, slightly or completely off-resonance. It can be shown that, under proper tuning of the excitation fields, the condition $\Omega_2 \gg \Omega_1\sqrt{N}$ assures neat selectivity in the JC model, where resonant Rabi oscillations will happen only inside the subspace $\mathcal{H}_N$. In this case, Eq. (1) turns into the selective Hamiltonian

$$\widehat{H}_N = \hbar\sqrt{N}\Omega\big(|g\rangle\langle e| \otimes |N\rangle\langle N-1| + |e\rangle\langle g| \otimes |N-1\rangle\langle N|\big), \quad (2)$$

describing the flip-flop interaction of two effective spin-1/2 systems, $\{|g\rangle, |e\rangle\}$ and $\{|N-1\rangle, |N\rangle\}$. Specific implementations have been proposed in microwave CQED [12]

and in trapped ions [13], but other systems, like an atom inside an optical cavity, or a superconducting qubit coupled to a coplanar waveguide resonator, can also enjoy a similar behavior.

From the unitary time evolution of the total density operator, $\dot{\hat{\rho}} = [\widehat{H}_N, \hat{\rho}]/i\hbar$, the first and second time derivatives of the expectation value of a time-independent probe operator $\hat{B}$ can be expressed as

$$\frac{d\langle\hat{B}\rangle}{dt} = \frac{1}{i\hbar}\langle[\hat{B}, \widehat{H}_N]\rangle, \tag{3}$$

$$\frac{d^2\langle\hat{B}\rangle}{dt^2} = \frac{1}{(i\hbar)^2}\langle\big[[\hat{B}, \hat{H}_N], \hat{H}_N\big]\rangle. \tag{4}$$

We study here a more general case, allowing decoherence in the (field) system but disregarding decoherence in the probe. We do that based on the fact that most physical setups use probes with long lifetimes compared to the ones of the systems to measure. Under this assumption, we consider the most general master equation $\dot{\hat{\rho}} = \mathcal{L}\hat{\rho}$ in the Lindblad form [18],

$$\dot{\hat{\rho}} = \frac{1}{i\hbar}\left[\hat{H}_N, \hat{\rho}\right] + \sum_m \kappa_m (\hat{A}_m\hat{\rho}\hat{A}_m^\dagger - \frac{1}{2}\hat{A}_m^\dagger\hat{A}_m\hat{\rho} - \frac{1}{2}\hat{\rho}\hat{A}_m^\dagger\hat{A}_m)\,, \tag{5}$$

where κ_m express decay rates and Lindblad operators A_m and $A_m^\dagger$ are associated with the (field) system.

We calculate first

$$\frac{d\langle\hat{B}\rangle}{dt} = \frac{1}{i\hbar}\langle[\hat{B}, \hat{H}_N]\rangle + \mathrm{Tr}\left[\sum_m \kappa_m (\hat{A}_m\hat{\rho}\hat{A}_m^\dagger - \frac{1}{2}\hat{A}_m^\dagger\hat{A}_m\hat{\rho} - \frac{1}{2}\hat{\rho}\hat{A}_m^\dagger\hat{A}_m)\hat{B}\right].$$

The second term on the r.h.s. can be rewritten under the trace as

$$\mathrm{Tr}\frac{1}{2}\left[\sum_m \kappa_m \hat{\rho}\left([\hat{A}_m^\dagger, \hat{B}]\hat{A}_m + \hat{A}_m^\dagger[\hat{B}, \hat{A}_m]\right)\right]$$

and, since $[\hat{B}, \hat{A}_m^\dagger] = [\hat{B}, \hat{A}_m] = 0$, we have

$$\frac{d\langle\hat{B}\rangle}{dt} = \frac{1}{i\hbar}\langle[\hat{B}, \hat{H}_N]\rangle. \tag{6}$$

In this expression, see similarity with Eq. (3), the dynamics of probe operator expectation value $\langle\hat{B}\rangle$ does not seem affected by the system decay. This is certainly not the case, as the time-dependent expectation value on the r.h.s. of Eq. (6) will in general be susceptible to decoherence: the calculation involves time-dependent $\rho(t)$ following Eq. (5). However, at $t = 0$ the noise terms vanish identically, i.e., the time derivative of any probe expectation value at $t = 0$ is independent of any field-decohering Lindblad environment. For this particular time, choosing a probe operator $\hat{B} = |e\rangle\langle e|$ and probe-system initial state $\rho(0) = |g\rangle\langle g| \otimes \rho_f$, we obtain

$$\left.\frac{dP_e(\tau)}{d\tau}\right|_{\tau=0} = 0, \tag{7}$$

with dimensionless time $\tau \equiv \Omega t$ and $P_e(\tau) \equiv \langle|e\rangle\langle e|\rangle$. For the second derivative, we obtain

$$\frac{d^2\langle\hat{B}\rangle}{dt^2} = \frac{1}{(i\hbar)^2}\langle\big[[\hat{B}, \hat{H}_N], \hat{H}_N\big]\rangle + \mathrm{Tr}\Bigg[\sum_m \kappa_m (\hat{A}_m\hat{\rho}\hat{A}_m^\dagger - \frac{1}{2}\hat{A}_m^\dagger\hat{A}_m\hat{\rho} - \frac{1}{2}\hat{\rho}\hat{A}_m^\dagger\hat{A}_m)[\hat{B}, \hat{H}_N]\Bigg]. \tag{8}$$

This is a different situation and the trace will not vanish in general, as before. However, considering again the observable $\hat{B} = |e\rangle\langle e|$ and probe-system initial state $\rho(0) = |g\rangle\langle g| \otimes \rho_f$, it follows that the second term in Eq. (8) vanishes again, and we have

$$\left.\frac{d^2P_e(\tau)}{d\tau^2}\right|_{\tau=0} = P_N. \tag{9}$$

Here, $P_N = \mathrm{Tr}[\rho_f|N\rangle\langle N|]$ is the probability of finding Fock state $|N\rangle$ in the initially unknown harmonic oscillator state.

Equation (9) describes a remarkable result, it shows that the curvature of the function $P_e(\tau)$, at vanishing $\tau = 0$, contains *undisturbed* information about P_N. This valuable field information is encoded correctly in the level statistics of the two-level probe even in presence of a field reservoir of a general kind. Needless to say, all previous results hold when the usual thermal bath is considered as a reservoir for the harmonic oscillator. At zero temperature, for example, Lindblad operators $\hat{A}_m$ and $\hat{A}_m^\dagger$ would have to be replaced by $\hat{a}$ and $\hat{a}^\dagger$, respectively, and κ would represent the decay rate of the single field mode. The counter-intuitive results of (6) and (9) are of an infinitesimal nature and, without harming their theoretical importance, should suffer high-order corrections when dealing with a discrete sampling of interaction times, as will be explained later.

In order to measure the complete field population, P_n, $\forall n$, one just needs to tune resonantly the other selected subspaces $\mathcal{H}_n$ and follow a similar procedure. The measurement of all P_n allows the estimation of the complete Wigner function $W(\alpha)$ of ρ, conditioned to the realization of previous arbitrary field displacements $D(-\alpha)$ in phase space. For that, we have to recall that the Wigner function can be expressed [1], among other possibilities, as

$$W(\alpha) \equiv 2\sum_{n=0}^{\infty}(-1)^n P_n(-\alpha), \tag{10}$$

where $P_n(-\alpha)$ stands for the field population after a displacement $D(-\alpha)$. Another rather original manner of

reconstructing the quantum field state in phase space is via the instantaneous measurement of the Q-function [1], defined as $Q = \langle\alpha|\rho_f|\alpha\rangle$. It can be shown that

$$\begin{aligned} Q(\alpha) &= \mathrm{Tr}[D(-\alpha)\rho_f D^\dagger(-\alpha)|0\rangle\langle 0|] = \mathrm{Tr}[\rho_f(-\alpha)|0\rangle\langle 0|] \\ &= P_0(-\alpha). \end{aligned} \tag{11}$$

This means that, following Eq. (9), measuring instantaneously the probability of having Fock state $|0\rangle$ after field displacements $D(-\alpha)$, amounts to a full measurement of the Q-function. Note that this will only require the tuning of a single selective subspace, reducing enormously the experimental efforts when compared to the case of the Wigner function. Typically, coherent displacements can be realized at a very high rate, depending mainly on the intensity of the excitation fields, so this is not a critical issue.

From a fundamental point of view, our proposal suggests that, no matter how short the lifetime of a certain system is, there would always exist the possibility of encoding its quantum information in two-level probe statistics at infinitesimal interaction times. In other words, measuring quantum states may not require long-living systems or strong probe-system coupling, against common belief. In this work, we have not proved this conjecture in general. However, we have given a particular example, the case of a quantum harmonic oscillator or a single mode field, that is applicable to many physical setups.

Turning to more practical considerations, we stress that our scheme employs exclusively, as a final readout mechanism, the measurement of the population of the excited state of a two-level probe at different times. This probe population is measured directly by ionization, in the case of cavity QED, or fluorescence, for trapped ions and atoms, with suitable additional fields, different from the ones used in Eq. (1). This is a standard measurement in many quantum-optical experimental setups of interest and has been realized routinely with high accuracy and efficiency [6, 9, 19].

Estimating derivatives from a finite-time experimental sampling will produce higher order corrections and an overhead in the number of repetition measurements. In consequence, the benefit of instantaneous measurements demands an improved measurement accuracy, which can be seen as follows. To determine the discrete second derivative in Eq. (9), $\ddot{P}_e(0)$, we measure P_e at $0, \tau, 2\tau$, for small τ, and calculate

$$\frac{P_e(2\tau) - 2P_e(\tau) + P_e(0)}{\tau^2} = \ddot{P}_e(0) + o(\tau), \tag{12}$$

The measurement results are scattered around each P_e with variance $\Delta_e^2 = \Delta_q^2 + \Delta_t^2$, referring to quantum-mechanical and technical noise, the latter arising from imperfections in preparation, timing, and measurement. For small values of τ and probe operator $\hat{B} = |e\rangle\langle e|$, we have $\Delta_q^2(\tau) = \langle\hat{B}^2\rangle - \langle\hat{B}\rangle^2 = P_e(\tau) - P_e(\tau)^2 \sim P_N\tau^2$, following the Taylor expansion of $P_e(\tau)$ around zero time. Performing M measurements at each of the three times of Eq. (12), we can reduce the uncertainty in the three expectation values by a factor $1/\sqrt{M}$. Thus, our estimate for the second derivative comes with an error variance

$$\Delta^2 \sim \frac{6P_N\tau^2 + 4\Delta_t^2}{M\tau^4}, \tag{13}$$

while the *signal* is $\sim P_N$. Then, in order to achieve a signal-to-noise ratio larger than unity, $P_N/\Delta > 1$, we need a number of measurements

$$M > \tau^{-2} + \Delta_t^2\tau^{-4}, \tag{14}$$

approximately. In this way, as we reduce τ by a factor f, in order to improve the approximation of $\ddot{P}_e(0)$, we need to increase the number of measurements by f^2 (or f^4 if technical errors dominate) to maintain the desired signal-to-noise ratio. Consequently, the total probe-system interaction time $M\tau$ summed over all measurements increases, while each measurement outcome represents the effect of an arbitrarily short interaction time τ. Although this is a high price to pay, contamination of the oscillator by decoherence can be kept arbitrarily small, no matter how fast it is.

Quantitatively, the determination of $\ddot{P}_e(0)$ is associated with an error $\sim \dddot{P}_e(0)\tau$ due to the finiteness of τ. In this case, we can use Eq. (5) to obtain an expression for $\dddot{P}_e(0)$ and observe two contributions: a unitary component, present even in the absence of noise, and an additional term $\propto \kappa/\Omega$ due to the noise [20]. In order to determine $\ddot{P}_e(0)$ to accuracy q, we need $\tau < q/\dddot{P}_e(0)$. This, in turn, requires $M > q^{-2}\dddot{P}_e(0)^2$ for ideal measurements and $M > q^{-4}\dddot{P}_e(0)^4\Delta_t^2$ in the case of technical-dominated errors. Hence, the number M of required measurements increases polynomially with κ (in the case $\kappa \gg \Omega$ like κ^2 resp. κ^4). Remarkably, no matter how strong κ is, we can always make sure that it does not affect the measurement result by shortening τ. This procedure finds its natural limit $\tau > \Omega/\omega$, where ω is the smallest dominant frequency of the specified dynamics ($\omega = \Omega_2^2/\Delta$ in our example, which implies $\tau > \Omega_1/\Omega_2$). Note that we only need to know an upper bound of the decoherence rate to perform an accurate measurement, while alternative methods involve specific decoherence models.

The proposed method can be applied to any physical system enjoying JC selective interactions, and may allow for the complete measurement of elusive fast-decaying systems, as is the case of intracavity optical fields, among others. Furthermore, since the scheme is based on infinitesimal transfer of information, this method could observe initial state conditions. For example, it could be used to test the initial purity of a system, as well as its unavoidable entanglement with external degrees of freedom at zero interaction time. This may prove useful for

testing the validity of usual assumptions taken in decoherence theories, such as system and environment being initially in a separable state or markovian dynamics.

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* Electronic address: msantos@fisica.ufmg.br
† Electronic address: giedke@mpq.mpg.de
‡ Electronic address: enrique.solano@physik.lmu.de

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